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The Diffraction Problem for a Half Plane with Different Face Impedances Revisited

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The problem of the diffraction of an electromagnetic wave by a half plane with different face impedances is dealt with, following a rigorous approach based on the $[L_2^+(\mathbb{R})]_2$ theory of systems of Wiener–Hopf equations with piecewise continuous presymbols. The corresponding operator is defined in spaces of physically admissible solutions, the Sobolev spaces $H_\alpha^+(\mathbb{R}) \times H_{\alpha-1}^+(\mathbb{R})$ for $\alpha > \frac{1}{2}$, and its Fredholm characteristics are determined. For $\frac{1}{2} < \alpha < 1$ it is shown that the operators are invertible and their inverses are calculated. In the final section the inverse of a related operator presented by Meister and Speck is also obtained.

1. INTRODUCTION

The problem of the scattering of a plane wave by a half plane with different face impedances was treated for the first time by Maliuzhinetz in 1959. Meanwhile it has been recognized that the corresponding boundary value problem for the Helmholtz equation leads to a pair of coupled Wiener–Hopf equations that remained unsolvable until 1975, when Hurd [4] introduced the so-called Wiener–Hopf–Hilbert method and exemplified its application with the problem referred to above. In 1981, Hurd and Przezdziecki [5] reexamined the problem more carefully and in 1984, Lüneburg and Hurd [6], using the ideas of Daniele [2] solved again the problem, now by a matrix function-theoretic factorization technique.

The aim of the present paper is to propose a new and rigorous approach to the study of the problem, using the theories available for systems of Wiener–Hopf equations in $[L_2(\mathbb{R}^+)]_n$ (see, for instance, [1, 8]). Specifically, we start by establishing the equivalence between the boundary value problem and a system of pseudodifferential equations. The solution is sought in the Sobolev-space $H_{1/2}^+(\mathbb{R}) \times H_{-1/2}^+(\mathbb{R})$, the space of physically admissible solutions (see [3]). However, once the problem is reduced to

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$[L_2^+(\mathbb{R})]_2$, it is seen that the symbol of the corresponding Wiener–Hopf operator is 2-singular (see, for instance, [1 or 8]), i.e., the operator so defined is not a Fredholm operator. However, this is not the case if we take the solutions in $H_\alpha^+(\mathbb{R}) \times H_{\alpha-1}^+(\mathbb{R})$, $n/2 < \alpha < (n+1)/2$ (n a positive integer) that are also spaces of physically admissible solutions. In this setting we obtain in Section 3 the equivalent operator in $[L_2^+(\mathbb{R})]_2$ and in the subsequent section we succeed in calculating the generalized factorization of the presymbol, thus obtaining in particular, for $\frac{1}{2} < \alpha < 1$, the inverse operator.

Another approach has been suggested by Meister and Speck in [7], modifying the original operator, by introducing what they call the compatibility condition, in order to obtain a non-singular symbol. In Section 5 we complete this procedure since we are able to perform the generalized canonical factorization of the presymbol. Finally we note that during the writing of this paper we received a preprint [11] in which general Sommerfeld diffraction problems with third kind boundary conditions are considered. However, in what concerns the present problem the authors only show that the modified Wiener–Hopf operator is Fredholm with zero index (see [11, Theorem 4.2]) leaving out the question of operator invertibility as well as the actual computation of the inverse.

2. Formulation of the Problem

Let us begin by considering the direct differential formulation of the diffraction problem for the half plane with different face impedances as a boundary value problem for the Helmholtz equation.

The edge of the screen is chosen to coincide with the z-axis and the half plane is located at $y=0$, $x \geqslant 0$. The incident electric field, $\mathbf{E}_i = \varphi_i \mathbf{e}_z$ ($\mathbf{e}_z$ denotes the z-unit vector), is assumed to be parallel to the edge of the half plane. The diffracted electric field will then be expressed in terms of a scalar potential $\varphi(x, y)$ by $\mathbf{E} = \varphi \mathbf{e}_z$. The function φ satisfies the Helmholtz equation

$$\Delta\varphi + k^2\varphi = 0, \qquad y \neq 0 \tag{2.1}$$

where k is the free space wave number. We suppose yet that the medium is dissipative, that is,

$$\operatorname{Im} k > 0 \tag{2.2}$$

Denoting by $s_j = k \sin \phi_j$, $j = 1, 2$, the face impedances with $0 < \phi_j < \pi/2$, φ must satisfy the impedance boundary conditions

$$\frac{\partial \varphi}{\partial y}(x, 0\pm) \pm is_j \varphi(x, 0\pm)$$

$$= -\frac{\partial \varphi_i}{\partial y}(x, 0) \mp is_j \varphi_i(x, 0), \qquad x > 0, \quad j = 1, 2. \tag{2.3}$$

The continuity relations are

$$\varphi(x, 0+) - \varphi(x, 0-) = 0, \qquad x < 0 \tag{2.4}$$

$$\frac{\partial \varphi}{\partial y}(x, 0+) - \frac{\partial \varphi}{\partial y}(x, 0-) = 0, \qquad x < 0 \tag{2.5}$$

and the radiation condition is

$$\lim_{|y| \to \infty} \varphi(x, y) = 0, \qquad x \in \mathbb{R}. \tag{2.6}$$

Remark. In what follows we shall always assume $s_1 \neq s_2$. Indeed, if $s_1 = s_2$ we are led to a scalar problem, already study in [7].

If φ and φ_i are distributions depending on the parameter y the following mathematical formulation of the problem arises:

PROBLEM 1. To determine a temperate distribution $\varphi(\cdot, y) \in \mathscr{S}'(\mathbb{R})$ depending on the real parameter y which satisfies the Helmholtz equation (2.1) for every $y \neq 0$ and verifies the boundary conditions (2.3) to (2.6), the limits being understood in the sense of distributions.

We are going to derive the equivalent Wiener–Hopf formulation for this problem. To this end let us introduce two auxiliary distributions φ_1^+ and φ_2^+ with supports contained in $\mathbb{R}^+$ and replace conditions (2.4) and (2.5) by

$$\varphi(x, 0+) - \varphi(x, 0-) = \varphi_1^+(x), \qquad \operatorname{supp} \varphi_1^+ \subset \mathbb{R}^+ \tag{2.7}$$

$$-i\frac{\partial \varphi}{\partial y}(x, 0+) + i\frac{\partial \varphi}{\partial y}(x, 0-) = \varphi_2^+(x), \qquad \operatorname{supp} \quad \varphi_2^+ \subset \mathbb{R}^+. \tag{2.8}$$

Using the Fourier transformation $\mathscr{F}$ in $\mathscr{S}'(\mathbb{R})$ it is easily seen that if φ is a solution of the Helmholtz equation its Fourier transform $\Phi(\cdot, y) = (\mathscr{F}_x \varphi)(\cdot, y)$ satisfies

$$\Phi(\omega, y) = \left[\pm\frac{1}{2} \Phi_1^+(\omega) + \frac{1}{2\beta(\omega)} \Phi_2^+(\omega) \right] e^{\pm i\beta(\omega) y}, \qquad y \gtrless 0, \tag{2.9}$$

where $\Phi_j^+ = \mathscr{F}\varphi_j^+$, $j=1, 2$, and

$$\beta(\omega) = (k^2 - \omega^2)^{1/2} \tag{2.10}$$

in which the branch is chosen so that $\operatorname{Im}\beta(\omega) > 0$ for $\omega \in \mathbb{R}$ in order to garantee that condition (2.6) is satisfied.

Now, using the impedance conditions (2.3) and letting f_j, $j=1, 2$, be distributions on $\mathbb{R}^+$ given by

$$f_j = r\left(-s_j\varphi_i(\cdot, 0) \pm i\frac{\partial\varphi_i}{\partial y}(\cdot, 0)\right), \qquad j=1, 2 \tag{2.11}$$

the following system of pseudodifferential equations is obtained,

$$r(\mathscr{F}^{-1}G * \boldsymbol{\varphi}^+) = \mathbf{f}, \tag{2.12}$$

where r denotes the operator of restriction to $\mathbb{R}^+$, $\boldsymbol{\varphi}^+ = [\varphi_1, \varphi_2]^{\mathrm{T}}$, $\mathbf{f} = [f_1, f_2]^{\mathrm{T}}$, and G is the matrix function

$$G = \frac{1}{2}\begin{bmatrix} s_1 + \beta & 1 + s_1/\beta \\ -(s_2 + \beta) & 1 + s_2/\beta \end{bmatrix} \tag{2.13}$$

which is already well known (see [2, 4–6]).

PROBLEM 2. Given a vector distribution $\mathbf{f}$ on $\mathbb{R}^+$ to determine a vector distribution $\boldsymbol{\varphi}^+ \in [\mathscr{S}'(\mathbb{R})]_2$ with $\operatorname{supp}\boldsymbol{\varphi}^+ \subset \mathbb{R}^+$ which satisfies the system of pseudodifferential equations (2.12).

It is easy to prove the following:

THEOREM 2.1. *Problems* 1 *and* 2 *are equivalent, i.e., if* φ *is a solution to Problem* 1 *then* $\boldsymbol{\varphi}^+$ *satisfying* (2.12) *is a solution to Problem* 2, *and, reciprocally, to each solution* $\boldsymbol{\varphi}^+$ *of Problem* 2 *there corresponds a unique* φ *that is a solution to Problem* 1 *and whose Fourier transform is given by* (2.9).

Having established the above theorem we shall now concentrate our attention on Problem 2. However, we have yet an excessively general setting for the analysis of existence and uniqueness of the solution to be feasible. So, we begin by looking for the natural functional spaces for the problem. Following [3], we want to determine the spaces of physically admissible solutions, i.e., spaces X of temperate (vector) distributions with support in $\mathbb{R}^+$ for which the energy associated with the potential φ corresponding to each element of X is finite in every bounded region Ω of $\mathbb{R}^2$ and such that if Eq. (2.12) has a solution in X, it is unique.

Using exactly the same procedure as in [3], starting from Eq. (2.9) and imposing the finite energy condition

$$\int_\Omega \left|\frac{\partial\varphi}{\partial x}\right|^2 dx\,dy < \infty, \qquad \int_\Omega \left|\frac{\partial\varphi}{\partial y}\right|^2 dy\,dx < \infty$$

we are led to the following conclusion: *the energy stored in the diffracted field is finite if and only if the boundary functions* φ_1^+ *and* φ_2^+ *belong to* $H^+_{1/2}(\mathbb{R})$ *and* $H^+_{-1/2}(\mathbb{R})$, *respectively.*

Notation. $H_\alpha(\mathbb{R})$, $\alpha\in\mathbb{R}$, denotes the usual Sobolev space of distributions ψ such that $\mathscr{F}\psi(\omega)=(\omega^2+1)^{-(\alpha/2)}F(\omega)$ for $F\in L_2(\mathbb{R})$, with $\|\psi\|_\alpha=\|F\|_{L_2}$. By $H^\pm_\alpha(\mathbb{R})$ we mean the closed subspaces of $H_\alpha(\mathbb{R})$ whose elements have supports in $\mathbb{R}^\pm$. We shall represent by $H_\alpha(\mathbb{R}^+)$ the subspace of $\mathscr{S}'(\mathbb{R}^+)$ whose elements f have an extension lf to $\mathbb{R}$ that belongs to $H_\alpha(\mathbb{R})$; this space is isomorphic to the quotient space $H_\alpha(\mathbb{R})/H^-_\alpha(\mathbb{R})$ endowed with the quotient norm. In what follows we shall also use the conventions: H_α, $H^\pm_\alpha$ stand for $H_\alpha(\mathbb{R})$ and $H^\pm_\alpha(\mathbb{R})$, respectively. When dealing with products of two equal spaces we shall write $[H_\alpha]_2$ instead of $H_\alpha\times H_\alpha$.

So, we begin by regarding Eq. (2.12) as defining the following pseudo-differential operator

$$\begin{aligned} &\mathscr{K}_{1/2}\colon H^+_{1/2}\times H^+_{-1/2}\to[H_{-1/2}(\mathbb{R}^+)]_2\\ &\mathscr{K}_{1/2}\boldsymbol{\varphi}^+=r(\mathscr{F}^{-1}G*\boldsymbol{\varphi}^+)\end{aligned} \tag{2.14}$$

which is easily seen to be a bounded linear operator for the usual product topology.

It is worthwhile noting that this operator is contained in the class considered by Speck, Hurd, and Meister in [11]. They have shown that for the case where G is the matrix considered above, the operator $\mathscr{K}_{1/2}$ is not Fredholm (indeed, its range is not closed). We shall concentrate on this question in the next sections, where we are able to find spaces of physically admissible solutions.

It is obvious that every subspace of $H^+_{1/2}\times H^+_{-1/2}$ is a space of finite energy. So, it makes sense to consider, for $\alpha>\frac{1}{2}$, the bounded linear operator

$$\begin{aligned} &\mathscr{K}_\alpha\colon H^+_\alpha\times H^+_{\alpha-1}\to[H_{\alpha-1}(\mathbb{R}^+)]_2\\ &\mathscr{K}_\alpha\boldsymbol{\varphi}^+=r(\mathscr{F}^{-1}G*\boldsymbol{\varphi}^+)\end{aligned} \tag{2.15}$$

and to investigate the Fredholm properties of $\mathscr{K}_\alpha$ in order to determine the values of α for which the domain of $\mathscr{K}_\alpha$ is a space of physically admissible solutions, i.e., for which the operator $\mathscr{K}_\alpha$ is injective or even invertible.

3. Reduction to $[L_2^+]_2$ and Fredholm Properties

We begin by showing that for real α the operator $\mathscr{K}_\alpha$ defined by (2.15) can be factorized in such a way that the equation

$$\mathscr{K}_\alpha \boldsymbol{\varphi}^+ = \mathbf{f} \tag{3.1}$$

with $\boldsymbol{\varphi}^+ \in H_\alpha^+ \times H_{\alpha-1}^+$, $\mathbf{f} \in [H_{\alpha-1}(\mathbb{R}^+)]_2$ can be replaced by an equivalent equation in $[L_2^+]_2$.

For this purpose, let us introduce some notation. We denote by $\mathscr{P}^\pm$ the orthogonal projection of $[L_2]_2$ onto $[L_2^\pm]_2$ defined by

$$\mathscr{P}^\pm \boldsymbol{\psi}(t) = \boldsymbol{\psi}(t)\, h(\pm t) \tag{3.2}$$

for almost every $t \in \mathbb{R}$, where h denotes the Heaviside unit step function. $\mathscr{D}_{\alpha\pm}$ will represent convolution operators with symbols $D_{\alpha\pm}$ defined by

$$\mathscr{D}_{\alpha+}: H_\alpha \times H_{\alpha-1} \to [L_2]_2 \tag{3.3}$$

$$\mathscr{D}_{\alpha-}: [L_2]_2 \to [H_{\alpha-1}]_2 \tag{3.4}$$

$$\mathscr{D}_{\alpha\pm} = \mathscr{F}^{-1} D_{\alpha\pm} \mathscr{F} \tag{3.5}$$

with

$$D_{\alpha+}(\omega) = (k+\omega)^{\alpha-1} \operatorname{diag}[(k+\omega)^2, 1] \tag{3.6}$$

$$D_{\alpha-}(\omega) = (k-\omega)^{-(\alpha-1)} I, \tag{3.7}$$

where I is the identity matrix and $(k \pm \omega)^{\pm(\alpha-1)}$ for a non-integer α have branch cuts $\Gamma_\pm = \{z \in \mathbb{C}\colon z = \mp(k + i\sigma),\ \sigma > 0\}$, respectively. It is readily seen that the operators so defined are invertible.

We have the following result:

Theorem 3.1. *For real α the operator $\mathscr{K}_\alpha$ defined by* (2.15) *admits the representation*

$$\mathscr{K}_\alpha = \mathscr{A}_{\alpha-} \mathscr{T}_\alpha \mathscr{A}_{\alpha+}, \tag{3.8}$$

where the operators $\mathscr{A}_{\alpha+}$ and $\mathscr{A}_{\alpha-}$ defined by

$$\mathscr{A}_{\alpha+} = \mathscr{D}_{\alpha+}|_{H_\alpha^+ \times H_{\alpha-1}^+}\colon H_\alpha^+ \times H_{\alpha-1}^+ \to [L_2^+]_2 \tag{3.9}$$

$$\mathscr{A}_{\alpha-} = r\, \mathscr{D}_{\alpha-}|_{[L_2^+]_2}\colon [L_2^+]_2 \to [H_{\alpha-1}(\mathbb{R}^+)]_2 \tag{3.10}$$

are invertible and the operator $\mathscr{T}_\alpha$,

$$\begin{aligned} &\mathscr{T}_\alpha\colon [L_2^+]_2 \to [L_2^+]_2 \\ &\mathscr{T}_\alpha \boldsymbol{\psi}^+ = \mathscr{P}^+(\mathscr{F}^{-1} G_\alpha * \boldsymbol{\psi}^+) \end{aligned} \tag{3.11}$$

is a Wiener-Hopf operator with presymbol G_α given by

$$G_\alpha = D_{\alpha-}^{-1} G\, D_{\alpha+}^{-1}. \tag{3.12}$$

Proof. Using (2.15) and denoting by $\mathscr{G}$: $H_\alpha \times H_{\alpha-1} \to [H_{\alpha-1}]_2$ the convolution operator defined by

$$\mathscr{G}\psi = \mathscr{F}^{-1} G \mathscr{F}\psi,$$

we have

$$\begin{aligned}
\mathscr{K}_\alpha &= r\mathscr{G}|_{H_\alpha^+ \times H_{\alpha-1}^+}\\
&= r\mathscr{D}_{\alpha-}(\mathscr{P}^+ + \mathscr{P}^-)\mathscr{D}_{\alpha-}^{-1}\mathscr{G}\mathscr{D}_{\alpha+}^{-1}\mathscr{D}_{\alpha+}|_{H_\alpha^+ \times H_{\alpha-1}^+}\\
&= \mathscr{A}_{\alpha-}\mathscr{P}^+\mathscr{D}_{\alpha-}^{-1}\mathscr{G}\mathscr{D}_{\alpha+}^{-1}\mathscr{A}_{\alpha+}\\
&= \mathscr{A}_{\alpha-}\mathscr{T}_\alpha\mathscr{A}_{\alpha+}.
\end{aligned}$$

It remains to prove that $\mathscr{A}_{\alpha-}$ and $\mathscr{A}_{\alpha+}$ are invertible operators. For $\mathscr{A}_{\alpha+}$ it is easily seen that

$$\mathscr{A}_{\alpha+}^{-1} = \mathscr{D}_{\alpha+}^{-1}|_{[L_2^+]_2}.$$

For $\mathscr{A}_{\alpha-}$ we shall show that

$$\mathscr{A}_{\alpha-}^{-1} = \mathscr{P}^+\mathscr{D}_{\alpha-}^{-1} l,$$

where l: $[H_{\alpha-1}(\mathbb{R}^+)]_2 \to [H_{\alpha-1}]_2$ is an extension operator. To prove the equality above we start by showing that $\mathscr{A}_{\alpha-}$ is injective. In fact, if $\boldsymbol{\varphi}^+ \in [L_2^+]_2$ is an element of $\operatorname{Ker}\mathscr{A}_{\alpha-}$ then there exists $\boldsymbol{\varphi}^- \in [H_{\alpha-1}^-]_2$ such that

$$\mathscr{D}_{\alpha-}\boldsymbol{\varphi}^+ = \boldsymbol{\varphi}^-$$

so

$$\boldsymbol{\varphi}^+ = \mathscr{D}_{\alpha-}^{-1}\boldsymbol{\varphi}^- = \mathbf{0},$$

since $\mathscr{D}_{\alpha-}^{-1}\boldsymbol{\varphi}^- \in [L_2^-]_2$ and $[L_2^+]_2 \cap [L_2^-]_2 = \{\mathbf{0}\}$.

Let $\mathbf{f}$ be any function in $[H_{\alpha-1}(\mathbb{R}^+)]_2$ and define $l\mathbf{f}$ as an extension of $\mathbf{f}$ belonging to $[H_{\alpha-1}]_2$. Then

$$r\mathscr{D}_{\alpha-}\boldsymbol{\psi}^+ = \mathbf{f}$$

if and only if there exists $\boldsymbol{\psi}^- \in [H_{\alpha-1}^-]_2$ such that

$$\mathscr{D}_{\alpha-}\boldsymbol{\psi}^+ = l\mathbf{f} + \boldsymbol{\psi}^-$$

from which it follows that

$$\psi^{+} = \mathscr{P}^{+}\mathscr{D}_{\alpha}^{-1} l\mathbf{f}$$

which gives the expression for the inverse of $\mathscr{A}_{\alpha-}$ since $\mathscr{D}_{\alpha-}^{-1} l\mathbf{f} \in [L_2]_2$ for any $l\mathbf{f} \in [H_{\alpha-1}]_2$. ∎

The following proposition is an immediate consequence of Theorem 3.1.

COROLLARY 3.2. *For real* α *let* $\mathscr{K}_\alpha$ *be the operator defined by* (2.15). *Then the equation* (3.1) *is equivalent to the equation in* $[L_2^+]_2$

$$\mathscr{T}_\alpha \psi^{+} = \mathscr{A}_{\alpha-}^{-1}\mathbf{f} \qquad (\mathbf{f} \in [H_{\alpha-1}(\mathbb{R}^{+})]_2) \tag{3.13}$$

with $\mathscr{T}_\alpha$ *defined by* (3.11), *in the sense that there exists a one-to-one correspondence between the solutions of* (3.1) *and* (3.13) *given by*

$$\psi^{+} = \mathscr{A}_{\alpha+}\varphi^{+}. \tag{3.14}$$

Hence we have reduced the study of the properties of the operator $\mathscr{K}_\alpha$ to the study of the Wiener–Hopf operator $\mathscr{T}_\alpha$ in $[L_2^+]_2$ with presymbol

$$G_\alpha(\omega) = \frac{1}{2}\left(\frac{\beta_-}{\beta_+}\right)^{2\alpha-1} \begin{bmatrix} 1+\dfrac{s_1}{\beta} & \dfrac{\beta_+}{\beta_-}\left(1+\dfrac{s_1}{\beta}\right) \\ -\left(1+\dfrac{s_2}{\beta}\right) & \dfrac{\beta_+}{\beta_-}\left(1+\dfrac{s_2}{\beta}\right) \end{bmatrix}, \tag{3.15}$$

where, for later convenience, we have denoted by $\beta_\pm$ the functions

$$\beta_\pm(\omega) = (k \pm \omega)^{1/2}$$

with branch cuts $\Gamma_\pm$. G_α is a continuous matrix function on $\mathbb{R}$ with finite but different limits at $\pm\infty$, henceforth denoted by $G_\alpha(\pm\infty)$, respectively. Then we may use some known results for Wiener–Hopf operators in $[L_2^+]_2$ with piecewise continuous presymbols on $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (see, for instance, [8]). In particular, defining the symbol $\tilde{G}_\alpha(\omega, \mu)$ on $\dot{\mathbb{R}} \times [0, 1]$ by

$$\tilde{G}_\alpha(\omega, \mu) = \begin{cases} G_\alpha(\omega), & \omega \in \mathbb{R} \\ [G_\alpha(-\infty) - G_\alpha(\infty)]\mu + G_\alpha(\infty), & \omega = \infty, \mu \in [0, 1] \end{cases} \tag{3.16}$$

from the results of [1, 8] it is easily shown that the Fredholm property of $\mathscr{T}_\alpha$ is equivalent to

$$\det \tilde{G}_\alpha(\omega, \mu) \neq 0, \qquad (\omega, \mu) \in \dot{\mathbb{R}} \times [0, 1] \tag{3.17}$$

which means that G_α is 2-non-singular.

THEOREM 3.3. *For real α let $\mathscr{T}_\alpha$ be the Wiener–Hopf operator defined by* (3.11) *with presymbol G_α given by* (3.15). *Then $\mathscr{T}_\alpha$ is a Fredholm operator if and only if*

$$\alpha \in \mathbb{R} \backslash \{r\colon r = \tfrac{n}{2},\ n \text{ integer}\}.$$

Proof. We want to prove that

$$\det \tilde{G}_\alpha(\omega, \mu) \neq 0 \qquad (\omega, \mu) \in \dot{\mathbb{R}} \times [0, 1]$$

if and only if α is a real number such that

$$\alpha \neq n/2,\ n \text{ integer}.$$

After some trivial computations we obtain

$$\det \tilde{G}_\alpha(\omega, \mu) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\beta_-(\omega)}{\beta_+(\omega)}\right)^{4\alpha-3}\left(1+\dfrac{s_1}{\beta(\omega)}\right)\left(1+\dfrac{s_2}{\beta(\omega)}\right), & \omega \in \mathbb{R} \\ \dfrac{1}{2}\,[(1-a)\mu+a][(1+a)\mu-a], & \omega = \infty,\ \mu \in [0, 1] \end{cases}$$

with $a = e^{i\pi(2\alpha-1)}$. It is immediate to conclude that $\det \tilde{G}_\alpha(\omega, \mu) \neq 0$ for $\omega \in \mathbb{R}$ (recall that $s_j = k \sin \phi_j$, $j = 1, 2$). For $\omega = \infty$ a straightforward computation shows that for $\mu \in [0, 1]$ we have

$$(1-a)\mu + a = 0 \Leftrightarrow \mu = \frac{1}{2}, \qquad \alpha = \frac{n}{2},\ n \text{ even}$$

$$(1+a)\mu - a = 0 \Leftrightarrow \mu = \frac{1}{2}, \qquad \alpha = \frac{n}{2},\ n \text{ odd}$$

which completes the proof. ▌

We note that by means of the factorization of the operator $\mathscr{K}_\alpha$ established in Theorem 3.1, the last result implies that $\mathscr{K}_\alpha$ is a Fredholm operator if and only if α is a real number such that

$$\alpha \neq n/2,\ n \text{ integer}.$$

In particular, taking $n=1$ we are led to the original operator $\mathscr{K}_{1/2}$ defined on $H^+_{1/2}\times H^+_{-1/2}$ (see (2.14)) which is not a Fredholm operator, according to what we have already mentioned and what has been proved in [11].

We shall see in the next section that, for $\frac{1}{2}<\alpha<1$, $\mathscr{K}_\alpha$ is indeed an invertible operator which, in particular, implies that all the spaces $H^+_\alpha\times H^+_{\alpha-1}$, for $\alpha>\frac{1}{2}$, are in fact spaces of physically admissible solutions (because $H^+_{\alpha_1}\times H^+_{\alpha_1-1}\subset H^+_{\alpha_2}\times H_{\alpha_2-1}$, $\alpha_1>\alpha_2$, and the injectivity property remains). However, for $\alpha\geqslant 1$, the operator $\mathscr{K}_\alpha$ will be seen not to be surjective or even Fredholm (in the countable number of cases determined above), and the defect number will be calculated.

4. Generalized Factorization and Invertibility of $\mathscr{K}_\alpha$

We begin this section by obtaining the inverse of the operator $\mathscr{K}_\alpha$, for $\frac{1}{2}<\alpha<1$, by means of Theorem 3.1 and the actual computation of the inverse of $\mathscr{T}_\alpha$ through the explicit calculation of the generalized canonical factorization of its presymbol G_α. Indeed, it follows from the general theory of Wiener–Hopf operators in $[L_2^+]_2$ with piecewise continuous presymbols that the invertibility of $\mathscr{T}_\alpha$ is equivalent to the existence of the generalized canonical factorization of G_α, i.e., the possibility of writing G_α as

$$G_\alpha=G_{\alpha-}\,G_{\alpha+}, \tag{4.1}$$

where

(1) $(1/\beta_+^2)G_{\alpha+}^{\pm 1}\in[\mathscr{F}(L_2^+)]_{2\times 2}$, $(1/\beta_-^2)G_{\alpha-}^{\pm 1}\in[\mathscr{F}(L_2^-)]_{2\times 2}$.

(2) The operator $\mathscr{U}_\alpha$ defined by

$$\mathscr{U}_\alpha\Phi=G_{\alpha+}^{-1}P^+G_{\alpha-}^{-1}\Phi \tag{4.2}$$

is a densily defined operator having a continuous extension to $[L_2]_2$ (hereafter denoted by the same symbol). Here

$$P^+=\mathscr{F}\mathscr{P}^+\mathscr{F}^{-1} \tag{4.3}$$

is the orthogonal projection of $[L_2]_2$ onto $[\mathscr{F}(L_2^+)]_2$.

Once the generalized canonical factorization is obtained, the inverse operator of $\mathscr{T}_\alpha$ is given by

$$\mathscr{T}_\alpha^{-1}=\mathscr{F}^{-1}\mathscr{U}_\alpha\mathscr{F} \tag{4.4}$$

(see [1, 8, 11]).

Given a piecewise continuous non-singular matrix function in $[L_\infty]_{2\times 2}$, except for the triangular case, no systematic procedure is known to obtain its generalized factorization. However, for a certain class of matrix functions, Daniele [2] has introduced a method to calculate a (function-theoretic) factorization that can possibly be, in many cases, a starting point to achieve the generalized factorization (see, for instance, [11]).

Although the presymbol G_α does not belong to the class considered by Daniele, it can be split into factors so that the Daniele method can still be used. In fact, we note that

$$G_\alpha = E_{\alpha -} D E_{\alpha +} \tag{4.5}$$

with

$$E_{\alpha -} = \frac{s_2 - s_1}{4 s_1 s_2} \beta_-^{2\alpha - 1} \begin{bmatrix} s_1 & s_1 \dfrac{1}{\beta_-} \\ s_2 & -s_2 \dfrac{1}{\beta_-} \end{bmatrix} \tag{4.6}$$

$$E_{\alpha +} = \beta_+^{1-2\alpha} \begin{bmatrix} 1 & 0 \\ 0 & \beta_+ \end{bmatrix} \tag{4.7}$$

and D is a matrix of Daniele's form given by

$$D = \begin{bmatrix} 1 & \dfrac{\gamma}{\beta_-} \\ \gamma \beta_- & 1 \end{bmatrix}, \tag{4.8}$$

where

$$\gamma = \frac{1}{s_2 - s_1} \left(s_1 + s_2 + \frac{2 s_1 s_2}{\beta} \right). \tag{4.9}$$

Applying the results of Daniele [2] and Rawlins [9] we split D in the form

$$D = D_l D_u, \tag{4.10}$$

where D_l, D_u are continuous matrix function on $\mathbb{R}$ holomorphically extendable into the lower/upper half plane, respectively, given by

$$D_{u,l} = (g_\pm)^{1/2} \begin{bmatrix} \cosh\left(\dfrac{1}{2}\beta_- t_\pm\right) & \dfrac{1}{\beta_-} \sinh\left(\dfrac{1}{2}\beta_- t_\pm\right) \\ \beta_- \sinh\left(\dfrac{1}{2}\beta_- t_\pm\right) & \cosh\left(\dfrac{1}{2}\beta_- t_\pm\right) \end{bmatrix}, \tag{4.11}$$

where $g_{\pm}$ are the Wiener–Hopf factors of the function g belonging to the extended Wiener algebra W,

$$g = g_{-} \, g_{+}, \tag{4.12}$$

defined by

$$g = \det D = 1 - \gamma^2 = -\frac{4 s_1 s_2}{(s_2 - s_1)^2} \left(1 + \frac{s_1}{\beta}\right)\left(1 + \frac{s_2}{\beta}\right) \tag{4.13}$$

(see [10] for the explicit formulas of $g_{\pm}$), and $t_{\pm}$ are the projections of t in the subalgebras $\mathring{W}^{\pm} = \mathscr{F}(L_1^{\pm})$, given by

$$t = \frac{1}{\beta} \ln \frac{1+\gamma}{1-\gamma} = t_{-} + t_{+}, \tag{4.14}$$

where for the logarithmic function we take any branch holomorphic in a region that contains the closure of the image of $(1+\gamma(\omega))(1-\gamma(\omega))^{-1}$ for real ω. Now we proceed to calculate the above additive decomposition of t. For convenience we write t in the form

$$t = \frac{1}{\beta_{-}} \ln\left(-\frac{s_2}{s_1}\right) + \frac{1}{\beta_{-}} \ln u \tag{4.15}$$

with

$$u = \frac{s_1 + \beta}{s_2 + \beta}. \tag{4.16}$$

Noting that t is a holomorphic function in the strip $|\operatorname{Im} z| < \operatorname{Im} k$ it follows that t_{+} is holomorphically extendable into the half plane $\operatorname{Im} z > -\operatorname{Im} k$ and

$$t_{+}(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{\lambda - z} \, t(\lambda) \, d\lambda, \qquad \operatorname{Im} z > -\operatorname{Im} k. \tag{4.17}$$

Next we establish a differential equation for t_{+}. Taking derivatives on both sides of the previous equation and integrating by parts, we obtain

$$\begin{aligned} \frac{d}{dz} t_{+}(z) &= -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{\lambda - z} \frac{1}{\beta_{-}^{3}(\lambda)} \ln u(\lambda) \, d\lambda + v_{+}(z) \\ &= \frac{1}{2\beta_{-}^{2}(z)} [t_{+}(z) - t_{+}(k)] + v_{+}(z), \qquad \operatorname{Im} z > -\operatorname{Im} k \end{aligned}$$

with

$$v_+ = -P^+ \left(\frac{1}{\beta_-} \frac{u'}{u}\right), \qquad \frac{u'}{u} = (s_1 - s_2)\frac{\lambda}{(s_1+\beta)(s_2+\beta)}.$$

Thus, in particular, for real ω, we have

$$t_+(\omega) = t_+(k) + \frac{1}{\beta_-(\omega)} \int_k^{\omega} \beta_-(\lambda)\, v_+(\lambda)\, d\lambda, \tag{4.18}$$

where the contour of integration is chosen so that it does not intersect the branch cut of β_-. Now, writing v_+ in the form

$$v_+ = P^+ \left(\frac{r_1}{\beta_+} + \frac{r_2}{\beta_-}\right),$$

where r_1 and r_2 are rational functions without poles on the real axis given by

$$r_1(z) = (s_2 - s_1)\frac{z(s_1 s_2 + \beta^2)}{\beta_-^2 (s_1^2 - \beta^2)(s_2^2 - \beta^2)},$$

$$r_2(z) = (s_1^2 - s_2^2)\frac{z}{(s^2 - \beta^2)(s_2^2 - \beta^2)}$$

and using the residue theorem, we obtain

$$v_+(\omega) = \frac{r_1(\omega)}{\beta_+(\omega)} + \sum_{j=0}^{4} \alpha_j \frac{1}{\omega - \lambda_j} \tag{4.19}$$

with

$$\begin{aligned} \lambda_0 = k, \qquad \lambda_{1,3} &= \pm(k^2 - s_1^2)^{1/2} = \pm k\cos\phi_1, \\ \lambda_{2,4} &= \pm(k^2 - s_2^2)^{1/2} = \pm k\cos\phi_2, \end{aligned} \tag{4.20}$$

and the constants α_j, $j = 0, ..., 4$ are such that the right-hand side of (4.19) is holomorphic in the upper half plane. Actually they are given by

$$\alpha_0 = \frac{s_1 - s_2}{s_1 s_2} \frac{k}{\beta_+(k)}, \qquad \alpha_j = (-1)^j \frac{1}{2\beta_-(\lambda_j)}, \quad j = 1, ..., 4.$$

Substituting these results in (4.18) and integrating we get

$$t_+(\omega) = t_+(k) + 2\sum_{j=0}^{4} \alpha_j + \frac{1}{\beta_-(\omega)} \ln u_1(\omega),$$

where

$$u_1 = \left(\frac{s_1 - \beta}{s_1 + \beta} \frac{s_2 + \beta}{s_2 - \beta} \prod_{j=1}^{4} \theta_j\right)^{1/2} \tag{4.21}$$

with

$$\theta_j(\omega) = \left(\frac{\beta_-(\lambda_j) - \beta_-(\omega)}{\beta_-(\lambda_j) + \beta_-(\omega)}\right)^{(-1)^j}, \qquad j = 1, ..., 4, \tag{4.22}$$

and for the logarithm and squareroot functions any consistent branch is taken.

Finally, as t belongs to $\mathring{W}$ we have $t_+(\omega) \to 0$ as $|\omega| \to \infty$ and so necessarily

$$t_+(k) = -2 \sum_{j=0}^{4} \alpha_j. \tag{4.23}$$

Then the desired projection is given by

$$t_+ = \frac{1}{\beta_-} \ln u_1 \tag{4.24}$$

The complementary projection t_- is simply obtained by using (4.14):

$$t_- = t - t_+ = -\frac{1}{\beta_-} \ln u_2 \tag{4.25}$$

with

$$u_2 = -\frac{s_1}{s_2} \frac{u_1}{u} = -\frac{s_1}{s_2} \frac{s_2 + \beta}{s_1 + \beta} u_1. \tag{4.26}$$

These results complete the computation of the factors D_u, D_l defined by (4.11). Substituting them into the cited expression we find

$$D_{u,l} = (g_\pm)^{1/2} \begin{bmatrix} cl_{1,2} & \dfrac{1}{\beta_-} sl_{1,2} \\ \beta_- sl_{1,2} & cl_{1,2} \end{bmatrix} \tag{4.27}$$

with

$$cl_{1,2} = \cosh(\pm \ln u_{1,2}) = \tfrac{1}{2} u_{1,2}^{-1/2} (u_{1,2} + 1) \tag{4.28}$$

$$sl_{1,2} = \sinh(\pm \ln u_{1,2}) = \pm \tfrac{1}{2} u_{1,2}^{-1/2} (u_{1,2} - 1). \tag{4.29}$$

In the next theorem we see that the splittings of G_α and D as in (4.5) and (4.10), respectively, furnish a generalized canonical factorization of G_α (for $\frac{1}{2} < \alpha < 1$).

THEOREM 4.1. *For* $\frac{1}{2} < \alpha < 1$ *let* G_α *be the matrix function defined by* (3.15). *Then the factorization*

$$G_\alpha = G_{\alpha -} G_{\alpha +} \tag{4.30}$$

with

$$G_{\alpha -} = E_{\alpha -} D_l = \frac{s_2 - s_1}{4 s_1 s_2} \beta_-^{2\alpha - 1} \left(\frac{g_-}{u_2} \right)^{1/2} \begin{bmatrix} s_1 & \dfrac{s_1}{\beta_-} \\ s_2 u_2 & -\dfrac{s_2}{\beta_-} u_2 \end{bmatrix} \tag{4.31}$$

$$G_{\alpha +} = D_u E_{\alpha +} = \frac{1}{2} \beta_+^{1-2\alpha} \left(\frac{g_+}{u_1} \right)^{1/2} \begin{bmatrix} u_1 + 1 & \dfrac{\beta_+}{\beta_-} (u_1 - 1) \\ \beta_-(u_1 - 1) & \beta_+(u_1 + 1) \end{bmatrix} \tag{4.32}$$

is a generalized canonical factorization.

Proof. We begin by noting that, using (4.6), (4.7), and (4.27), we get

$$G_{\alpha -} = E_{\alpha -} D_l = \frac{s_2 - s_1}{4 s_1 s_2} \beta_-^{2\alpha - 1} (g_-)^{1/2} \begin{bmatrix} s_1(cl_2 + sl_2) & \dfrac{s_2}{\beta_-} (cl_2 + sl_2) \\ s_2(cl_2 - sl_2) & -\dfrac{s_2}{\beta_-} (cl_2 - sl_2) \end{bmatrix} \tag{4.33}$$

and

$$G_{\alpha +} = D_u E_{\alpha +} = \beta_+^{1-2\alpha} (g_+)^{1/2} \begin{bmatrix} cl_1 & \dfrac{\beta_+}{\beta_-} sl_1 \\ \beta_- sl_1 & \beta_+ cl_1 \end{bmatrix} \tag{4.34}$$

from which (4.31) and (4.32) follow, using (4.28) and (4.29). In order to garantee that $G_\alpha = G_{\alpha -} G_{\alpha +}$ is indeed a generalized canonical factorization we have to verify conditions (1) and (2) of the definition (see the beginning of this section). A straightforward computation gives

$$G_{\alpha-}^{-1} = \frac{2}{s_2 - s_1}\beta_-^{1-2\alpha}(g_- u_2)^{-1/2}\begin{bmatrix} s_2 u_2 & s_1 \\ s_2\beta_- u_2 & -s_1\beta_- \end{bmatrix} \tag{4.35}$$

and

$$G_{\alpha+}^{-1} = \frac{1}{2}\beta_+^{2\alpha-1}(g_+ u_1)^{-1/2}\begin{bmatrix} 1+u_1 & \dfrac{1}{\beta_-}(1-u_1) \\ \dfrac{\beta_-}{\beta_+}(1-u_1) & \dfrac{1}{\beta_+}(1+u_1) \end{bmatrix}. \tag{4.36}$$

Now, noting that $u_{1,2}^{\pm 1} \in L_\infty$ and that $(g_\pm^{\pm 1} - g_\pm^{\pm}(\infty)) \in \mathring{W}^{\pm} \subset L_\infty$ it follows that, for $\frac{1}{2} < \alpha < 1$, $(1/\beta_\pm^2) G_{\alpha\pm}^{\pm 1} \in [\mathscr{F}(L_2)]_{2\times 2}$. On the other hand, the functions cl_1, $\beta_-^{\pm 1} sl_1$ are holomorphically extendable into the upper half plane and $cl_2 \pm sl_2$ are holomorphically extendable into the lower half plane. Then using the Paley–Wiener theorem it is easily concluded that

$$\frac{1}{\beta_+^2} G_{\alpha+}^{\pm 1} \in [\mathscr{F}(L_2^+)]_{2\times 2}, \qquad \frac{1}{\beta_-^2} G_{\alpha-}^{\pm 1} \in [\mathscr{F}(L_2^-)]_{2\times 2} \tag{4.37}$$

which completes the verification of condition (1). It remains to prove that condition (2) is satisfied. In fact we shall show that the linear operator $\mathscr{U}_\alpha$ defined on the whole of $[L_2]_2$ is continuous. To this end let $\mathbf{\Phi} = [\Phi_1, \Phi_2]^{\mathrm{T}}$ be an arbitrary element of $[L_2]_2$ and write

$$G_{\alpha-}^{-1}\mathbf{\Phi} = \begin{bmatrix} \beta_-^{1-2\alpha}\psi_1 \\ \beta_-^{2-2\alpha}\psi_2 \end{bmatrix}$$

with $\boldsymbol{\psi} = [\psi_1, \psi_2]^{\mathrm{T}}$ given by

$$\psi_{1,2} = \frac{2}{s_2 - s_1}(g_- u_2)^{-1/2}(s_2 u_2 \Phi_1 \pm s_1 \Phi_2)$$

belongs to $[L_2]_2$, by the above-cited properties of $u_{1,2}$ and $g_\pm$. Also, using these properties, we can write

$$\begin{aligned} \mathscr{U}_\alpha\mathbf{\Phi} &= G_{\alpha-}^{-1} P^+ G_{\alpha-}^{-1}\mathbf{\Phi} \\ &= \begin{bmatrix} \beta_+^{-1} l_1 \beta_+^{2\alpha-1} P^+ \beta_-^{1-2\alpha}\psi_1 + l_2\beta_+^{2\alpha-2} P^+ \beta_-^{2-2\alpha}\psi_2 \\ l_2\beta_+^{2\alpha-1} P^+ \beta_-^{1-2\alpha}\psi_1 + l_1\beta_+^{2\alpha-2} P^+ \beta_-^{2-2\alpha}\psi_2 \end{bmatrix} \end{aligned}$$

with $l_1, l_2 \in L_\infty$ given by

$$l_1 = \frac{1}{2}(1+u_1)(g_+u_1)^{-1/2}, \qquad l_2 = \frac{\beta_+}{2\beta_-}(1-u_1)(g_+u_1)^{-1/2}$$

Then, defining the operator $\mathscr{R}_v \colon L_2 \to L_2$ by

$$\mathscr{R}_v = \beta_+^v P^+ \beta_-^{-v} \mathscr{I}, \qquad -1 < v < 1,$$

with $\mathscr{I}$ denoting the identity operator on $[L_2]_2$, the continuity of $\mathscr{U}_\alpha$ ($\frac{1}{2} < \alpha < 1$) follows directly from the continuity of $\mathscr{R}_v$ (see [8, Theorem II.3.2] and recall the Plemelj formulas). ∎

As an immediate consequence of this theorem, we have:

COROLLARY 4.2. *Let* $\frac{1}{2} < \alpha < 1$. *Then*

(i) *The Wiener–Hopf operator* $\mathscr{T}_\alpha$ *defined by* (3.11) *is invertible with inverse defined by*

$$\mathscr{T}_\alpha^{-1}\psi = \mathscr{F}^{-1} G_{\alpha+}^{-1} P^+ G_{\alpha-}^{-1} \mathscr{F}\psi, \qquad \psi \in [L_2^+]_2,$$

where $G_{\alpha\pm}^{-1}$ *are given by* (4.36) *and* (4.35), *respectively.*

(ii) *The operator* $\mathscr{K}_\alpha$ *defined by* (2.15) *is invertible, with inverse given by*

$$\mathscr{K}_\alpha^{-1}\mathbf{f} = \mathscr{F}^{-1} D_{\alpha+}^{-1} G_{\alpha+}^{-1} P^+ G_{\alpha-}^{-1} P^+ D_{\alpha-}^{-1} \mathscr{F} l\mathbf{f}, \quad \mathbf{f} \in [H_{\alpha-1}(\mathbb{R}^+)]_2, \tag{4.38}$$

where $D_{\alpha\pm}$ *are the matrix functions defined by* (3.6) *and* (3.7), *respectively.*

Proof. (i) This result follows directly from the definition of $\mathscr{T}_\alpha$ and (4.2), (4.3), and (4.4).

(ii) Using Theorem 3.1, we have

$$\begin{aligned}\mathscr{K}_\alpha &= \mathscr{A}_{\alpha+}^{-1} \mathscr{T}_\alpha^{-1} \mathscr{A}_{\alpha-}^{-1} \\ &= \mathscr{D}_{\alpha+}^{-1} \mathscr{T}_\alpha^{-1} \mathscr{P}^+ \mathscr{D}_{\alpha-}^{-1} l.\end{aligned}$$

Then (4.38) is readily obtained through Eq. (3.5) to (3.7). ∎

Finally, we can complete our analysis for the operator $\mathscr{K}_\alpha$, for all real values of α, by showing tht the nullity and defect numbers of $\mathscr{K}_\alpha$ are given by

$$\begin{aligned}&\dim \operatorname{Ker} \mathscr{K}_\alpha = \max\{0, -n\} \\ &\operatorname{codim} \operatorname{Im} \mathscr{K}_\alpha = \max\{0, n\}, \qquad \alpha \in \left]\frac{n}{2}, \frac{n+1}{2}\right[, \ n \text{ integer},\end{aligned} \tag{4.39}$$

so that the Fredholm index is

$$\operatorname{ind} \mathscr{K}_\alpha = -n.$$

This can easily be seen by constructing the (possibly non-canonical) generalized factorization of G_α (see [1, 8]). In fact, if G_α admits such a factorization, i.e., if G_α can be written as

$$G_\alpha = G_{\alpha -} \Lambda G_{\alpha +} \tag{4.40}$$

with $G_{\alpha\pm}$ satisfying conditions (1) and (2) of the canonical factors (see the beginning of this section) and

$$\Lambda = \operatorname{diag}\left[\left(\frac{\beta_-}{\beta_+}\right)^{2\mu_1}, \left(\frac{\beta_-}{\beta_+}\right)^{2\mu_2}\right], \quad \mu_1, \mu_2 \text{ integers}, \mu_1 > \mu_2. \tag{4.41}$$

It is well known that

$$\dim \operatorname{Ker} \mathscr{T}_\alpha = \sum_{\mu_j < 0} |\mu_j| \tag{4.42}$$

$$\operatorname{codim} \operatorname{Im} \mathscr{T}_\alpha = \sum_{\mu_j > 0} \mu_j \tag{4.43}$$

and the Fredholm index of $\mathscr{T}_\alpha$ is the symmetric of the total index μ of the factorization given by $\mu = \mu_1 + \mu_2$.

So, for any real $\alpha \neq n/2$, n integer, let

$$\alpha = \alpha_0 + \frac{n}{2}, \qquad \alpha_0 \in \left]\frac{1}{2}, 1\right[. \tag{4.44}$$

Then, using (3.15), we have

$$G_\alpha = \left(\frac{\beta_-}{\beta_+}\right)^n G_{\alpha_0}.$$

Starting from the generalized canonical factorization of G_{α_0} given by (4.31) and (4.32) (with α replaced by α_0) we obtain the generalized factorization of G_α as follows:

— If n is even, take

$$G_{\alpha\pm} = G_{\alpha_0 \pm}, \qquad \Lambda = \left(\frac{\beta_-}{\beta_+}\right)^n I. \tag{4.45}$$

— If n is odd, using the splitting

$$\frac{\beta_-}{\beta_+} I = C_- \Lambda_1 C_+$$

with

$$C_- = \operatorname{diag}\left[\frac{1}{\beta_-}, \beta_-\right], \qquad C_+ = \operatorname{diag}\left[\beta_+, \frac{1}{\beta_+}\right],$$
$$\Lambda_1 = \operatorname{diag}\left[\left(\frac{\beta_-}{\beta_+}\right)^2, 1\right], \tag{4.46}$$

take

$$G_{\alpha_-} = G_{\alpha_0 -} C_-, \qquad G_{\alpha_+} = C_+ G_{\alpha_0 +} \tag{4.47}$$

$$\Lambda = \left(\frac{\beta_-}{\beta_+}\right)^{n-1} \Lambda_1 = \operatorname{diag}\left[\left(\frac{\beta_-}{\beta_+}\right)^{n+1}, \left(\frac{\beta_-}{\beta_+}\right)^{n-1}\right]. \tag{4.48}$$

Using the same procedure as before, the reader may convince himself that (4.40), with the factors defined as above, represents indeed a generalized factorization.

Moreover, from (4.42), (4.43) and the equivalence between the operators $\mathscr{T}_\alpha$ and $\mathscr{K}_\alpha$, established in Theorem 3.1, the nullity and defect numbers of $\mathscr{K}_\alpha$ (4.39) are immediately obtained.

To finish this section it is appropriate to note that as a consequence of the above analysis it is clear that all the spaces $H_\alpha^+ \times H_{\alpha-1}^+$ are physically admissible spaces for the operator $\mathscr{K}_\alpha$ if α is of the form (4.44) with n a non-negative integer.

5. On the Invertibility of an Operator Related to $\mathscr{K}_{1/2}$

In the previous sections the equivalence between the differential formulation of the diffraction problem for the half plane with different face impedances and its integral formulation (Theorem 2.1) was proved and the natural functional spaces for the problem were derived ($H_{1/2}^+ \times H_{-1/2}^+$ for the solutions and $[H_{-1/2}(\mathbb{R}^+)]_2$ for the excitations). However, as was shown, the corresponding operator $\mathscr{K}_{1/2}$ defined on these spaces yields an equivalent operator $\mathscr{T}_{1/2}$ on $[L_2^+]_2$ with a 2-singular presymbol $G_{1/2}$ (see (3.15) and Theorem 3.3), i.e., $\mathscr{K}_{1/2}$ is not a Fredholm operator. Our approach was to take the nearest possible spaces ($H_\alpha^+ \times H_{\alpha-1}^+$ with $\alpha > \frac{1}{2}$), where the energy condition is (still) verified and to study the operators $\mathscr{K}_\alpha$ in order to obtain their Fredholm characterization by means of the generalized factorization of the presymbol of the equivalent operators $\mathscr{T}_\alpha$.

As we have already mentioned, Meister and Speck [7] (see also [11]) have used a different approach consisting essentially in reducing the allowed excitation spaces for the problem and obtaining in this way an operator related to $\mathscr{K}_{1/2}$ which they prove in [11] to be Fredholm with zero index. For the sake of completeness we show in Theorem 5.1 how to obtain this operator. Afterwards in Theorem 5.2 we establish the invertibility property for the operator and determine its inverse.

To this end, let us begin by rewriting Eq. (2.12) in a more convenient form, by adding and subtracting them:

$$r(\mathscr{F}^{-1}G' * \boldsymbol{\varphi}^+) = \mathbf{f}' \tag{5.1}$$

with

$$\mathbf{f}' = \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} = \begin{bmatrix} -pr(\varphi_i(\cdot, 0)) \\ r\left(-q\varphi_i(\cdot, 0) + i\dfrac{\partial \varphi_i}{\partial y}(\cdot, 0)\right) \end{bmatrix} \tag{5.2}$$

for $p = \frac{1}{2}(s_1 + s_2)$, $q = \frac{1}{2}(s_1 - s_2)$, and

$$G' = \begin{bmatrix} q & 1 + \dfrac{p}{\beta} \\ p + \beta & \dfrac{q}{\beta} \end{bmatrix}. \tag{5.3}$$

The left-hand side of Eq. (5.1) defines a pseudodifferential operator $\mathscr{K}'_{1/2}$ on $H^+_{1/2} \times H^+_{-1/2}$ into $[H_{-1/2}(\mathbb{R}^+)]_2$, corresponding to $\mathscr{K}_{1/2}$ and thus a non-Fredholm operator. However, we have the following result (see [7, 11]):

THEOREM 5.1. *Let $f'_1 \in H_{-1/2}(\mathbb{R}^+)$ be such that its zero extension $l_0 f'_1$ to $\mathbb{R}$ belongs to $H^+_{-1/2}$ and consider the invertible convolution operators*

$$\begin{aligned} &\mathscr{A}\colon H_{-1/2} \to H_{1/2} && \mathscr{A} = \mathscr{F}^{-1}\frac{1}{\beta}\mathscr{F} \\ &\mathscr{A}_+ : L_2 \to H_{1/2} && \\ &\mathscr{A}_- : H_{-1/2} \to L_2 && \mathscr{A}_\pm = \mathscr{F}^{-1}\frac{1}{\beta_\pm}\mathscr{F}. \end{aligned} \tag{5.4}$$

Then the system of pseudodifferential equations (5.1) *has a unique solution given by*

$$\boldsymbol{\varphi}^+ = \begin{bmatrix} \varphi_1^+ \\ \varphi_2^+ \end{bmatrix} = \begin{bmatrix} \mathscr{A}_+ \psi_1^+ \\ l_0 f'_1 - q\mathscr{A}_+ \psi_1^+ - p\psi_2^+ \end{bmatrix}, \tag{5.5}$$

where $\boldsymbol{\psi}^+ = [\psi_1^+, \psi_2^+]^{\mathrm{T}} \in [L_2^+]_2$ *is the unique solution of the system of Wiener–Hopf equations*

$$\mathscr{P}^+(\mathscr{F}^{-1}G_M * \boldsymbol{\psi}^+) = \mathbf{f}^+ \tag{5.6}$$

for

$$\mathbf{f}^+ = \begin{bmatrix} \mathbf{f}_1^+ \\ \mathbf{f}_2^+ \end{bmatrix} = \begin{bmatrix} \mathscr{P}^+ \mathscr{A} l_0 f_1' \\ \mathscr{P}^+ \mathscr{A}_- l f_2' \end{bmatrix}, \tag{5.7}$$

where lf_2' *is any extension to* $\mathbb{R}$ *in* $H_{-1/2}$ *of* f_2', *and*

$$G_M = \begin{bmatrix} \dfrac{q}{\beta\beta_+} & 1 + \dfrac{p}{\beta} \\ 1 + \dfrac{p}{\beta} & \dfrac{q}{\beta_-} \end{bmatrix}. \tag{5.8}$$

Proof. Suppose that for a given $\mathbf{f}'$ verifying the hypothesis a solution $\boldsymbol{\varphi}^+ = [\varphi_1^+, \varphi_2^+]^{\mathrm{T}}$ of (5.1) exists. The first of Eq. (5.1) can be written as

$$q\varphi_1^+ + \varphi_2^+ + p\mathscr{A}\varphi_2^+ = lf_1' + \psi_1^-,$$

where $\psi_1^- \in H_{-1/2}^-$ and lf_1' denotes any extension to $\mathbb{R}$ of f_1'. But, by hypothesis $l_0 f_1'$ belongs to $H_{-1/2}^+$. Then, multiplying both members of the above equation by the unit step function h, we get

$$q\varphi_1^+ + \varphi_2^+ + p\mathscr{P}^+\mathscr{A}\varphi_2^+ = l_0 f_1' \tag{5.9}$$

or, by applying the invertible operator $\mathscr{A}$,

$$q\mathscr{A}\varphi_1^+ + \mathscr{A}\varphi_2^+ + p\mathscr{A}\mathscr{P}^+\mathscr{A}\varphi_2^+ = \mathscr{A}l_0 f_1', \tag{5.10}$$

where each term belongs to $H_{1/2}$. Now, for

$$\varphi_1^+ = \mathscr{A}_+\psi_1^+, \qquad \psi_1^+ \in L_2^+,$$

we can write (5.10) in the equivalent form

$$q\mathscr{A}\mathscr{A}_+\psi_1^+ + \mathscr{A}\varphi_2^+ + p\mathscr{A}\mathscr{P}^+\mathscr{A}\varphi_2^+ = \mathscr{A}l_0 f_1'. \tag{5.11}$$

Applying $\mathscr{P}^+$ to both members of the last equation, we obtain

$$\mathscr{P}^+[q\mathscr{A}\mathscr{A}_+\psi_1^+ + \psi_2^+ + p\mathscr{A}\psi_2^+] = \mathscr{P}^+\mathscr{A}l_0 f_1' = \mathbf{f}_1^+ \tag{5.12}$$

with

$$\psi_2^+ = \mathscr{P}^+\mathscr{A}\varphi_2^+ \in L_2^+. \tag{5.13}$$

The second of the Eq. (5.1) can be written as

$$(p+\mathscr{A}^{-1})\,\mathscr{A}_+\,\psi_1^+ + q\psi_2^+ = lf_2' + \psi_2^-, \tag{5.14}$$

where lf_1', $\psi_2^- \in H_{-1/2}$, and $\operatorname{supp}\psi_2^- \subset \mathbb{R}^-$. Applying the invertible operator $\mathscr{A}_-$ to both members of the above equation, we get

$$\mathscr{A}_-(p+\mathscr{A}^{-1})\,\mathscr{A}_+\,\psi_1^+ + q\mathscr{A}_-\,\psi_2^+ = \mathscr{A}_-\,lf_2' + \mathscr{A}_-\,\psi_2^-. \tag{5.15}$$

Noting that $\mathscr{A}_-\,\psi_2^- \in L_2^-$, applying $\mathscr{P}^+$ we obtain finally

$$\mathscr{P}^+[\mathscr{A}(p+\mathscr{A}^{-1})\,\mathscr{A}_+\,\psi_1^+ + q\mathscr{A}_-\,\psi_2^+] = \mathscr{P}^+\mathscr{A}_-\,lf_2' = \mathbf{f}_2^+. \tag{5.16}$$

It is immediately seen, by using the Fourier transformation, that (5.12) and (5.16) are the Eq. (5.6). So we have proved that each solution $\boldsymbol{\varphi}^+$ of (5.1) provides a solution $\boldsymbol{\psi}^+ = [\psi_1^+, \psi_2^+]^{\mathrm{T}} \in [L_2^+]_2$ of (5.6). By back substitution it is easy to verify that any solution $\boldsymbol{\psi}^+$ of (5.6) gives, through (5.5), a solution $\boldsymbol{\varphi}^+$ of (5.1).

In Theorem 5.2 we will see that (5.6) defines an invertible operator. So, this means, together with the above considerations, that for $\mathbf{f}'$ satisfying the hypothesis of this theorem the system (5.1) is uniquely solvable, which terminates the proof. ▮

The left-hand side of Eq. (5.6) defines the Wiener–Hopf operator of Meister and Speck [7],

$$\begin{aligned} &\mathscr{T}_M\colon [L_2^+]_2 \to [L_2^+]_2 \\ &\mathscr{T}_M\boldsymbol{\psi}^+ = \mathscr{P}^+(\mathscr{F}^{-1}G_M * \boldsymbol{\psi}^+) \end{aligned} \tag{5.17}$$

with presymbol G_M given by (5.8), which is a continuous matrix function in $\dot{\mathbb{R}}$. It can be seen that G_M is non-singular and, moreover, it has been proved in [11] that $\mathscr{T}_M$ is a Fredholm operator with zero index. Indeed, it is an invertible operator, as we are going to prove by deriving the generalized canonical factorization of G_M. Let

$$G_M = g_1 D_M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.18}$$

with

$$D_M = \begin{bmatrix} 1 & \dfrac{\chi}{\beta_+} \\ \beta_+\chi & 1 \end{bmatrix}, \tag{5.19}$$

where the functions g_1, $\chi \in W$ are given by

$$g_1 = 1 + \frac{p}{\beta}, \qquad \chi = \frac{q}{p+\beta}. \tag{5.20}$$

Note that (5.18) makes clear the analogy between G_M and the matrix function D defined by (4.8). Indeed, the middle factor D_M is formally identical to D. So, exactly by the same procedure followed in Section 4, the splitting of D_M as

$$D_M = D_{Ml} D_{Mu} \tag{5.21}$$

with $D_{Mu,l}$ holomorphically extendable into the upper/lower half plane, respectively, given by Daniele's method, can be performed (see Section 4). Doing this, we have

$$D_{u,l} = (\tilde{g}_\pm)^{1/2} \begin{bmatrix} cl_{1,2} & \dfrac{1}{\beta_+} sl_{1,2} \\ \beta_+ sl_{1,2} & cl_{1,2} \end{bmatrix}, \tag{5.22}$$

where $g_\pm$ are the Wiener–Hopf factors of $g \in W$ defined by

$$\tilde{g} = \det D_M = 1 - \chi^2 = -\frac{q^2}{s_1 s_2} \frac{g}{g_1^2}$$

with g given by (4.13) and

$$cl_{1,2} = \cosh(\pm \ln u_{1,2}) = \tfrac{1}{2} u_{1,2}^{-1/2}(u_{1,2}+1)$$
$$sl_{1,2} = \sinh(\pm \ln u_{1,2}) = \pm \tfrac{1}{2} u_{1,2}^{-1/2}(u_{1,2}-1)$$

with $u_{1,2}$ given by

$$u_1 = \frac{s_1+\beta}{s_2+\beta} u_2 \tag{5.23}$$

$$u_2 = \left(\frac{s_1+\beta}{s_1-\beta} \frac{s_2-\beta}{s_2+\beta} \prod_{j=1}^{4} \left(\frac{\beta_+(\lambda_j)+\beta_+}{\beta_+(\lambda_j)-\beta_+} \right)^{(-1)^j} \right)^{1/2} \tag{5.24}$$

for λ_j, $j = 1, ..., 4$, as in (4.20). This (function-theoretic) factorization yields the following factorization of G_M:

$$G_M = G_{M-} G_{M+} \tag{5.25}$$

where,

$$G_M = g_1 \quad D_{Ml} = \frac{1}{2} g_1 \left(\frac{g}{u_2}\right)^{1/2} \begin{bmatrix} 1+u_2 & \frac{1}{\beta_+}(1-u_2) \\ \beta_+(1-u_2) & 1+u_2 \end{bmatrix}, \tag{5.26}$$

$$G_{M+} = g_{1+} \, D_{Mu} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{2} g_{1+} \left(\frac{g_+}{u_1}\right)^{1/2} \begin{bmatrix} \frac{1}{\beta_+}(u_1-1) & u_1+1 \\ u_1+1 & \beta_+(u_1-1) \end{bmatrix}. \tag{5.27}$$

with $g_{1\pm}$ denoting the Wiener–Hopf factors of $g_1 \in W$ (see [10]). Moreover, the inverse factors are

$$G_{M-}^{-1} = \frac{1}{2} g_{1-}^{-1} (g_- u_2)^{-1/2} \begin{bmatrix} u_2+1 & \frac{1}{\beta_+}(u_2-1) \\ \beta_+(u_2-1) & u_2+1 \end{bmatrix} \tag{5.28}$$

$$G_{M+}^{-1} = \frac{1}{2} g_{1+}^{-1} (g_+ u_1)^{-1/2} \begin{bmatrix} \beta_+(1-u_1) & 1+u_1 \\ 1+u_1 & \frac{1}{\beta_+}(1-u_1) \end{bmatrix}. \tag{5.29}$$

Note that $g_{\pm}^{\pm 1}$, $g_{1\pm}^{-1}$ belong to $W^{\pm} \subset L_\infty$ and $u_{1,2}$ defined by (5.23) and (5.24) are such that $u_{1,2}^{\pm 1} \in L_\infty$. Furthermore, we have

$$u_{1,2} - 1 = O(|\omega|^{-1/2}) \qquad \text{as} \quad |\omega| \to +\infty$$

so that the matrix functions $G_{M\pm}^{\pm 1}$ belong to $[L_\infty]_{2\times 2}$. But then the representation (5.25) defines a generalized canonical factorization of G_M.

Having obtained a generalized canonical factorization of the presymbol G_M the following theorem, characterizing $\mathcal{T}_M$ as an invertible operator and providing its inverse, can be stated (see [1, 8]):

THEOREM 5.2. *Let $\mathcal{T}_M$ be the operator defined by* (5.17) *with presymbol G_M given by* (5.8). *Then $\mathcal{T}_M$ is an invertible operator with inverse*

$$\mathcal{T}_M^{-1} \psi^+ = \mathcal{F}^{-1} G_{M+}^{-1} P^+ G_{M-}^{-1} \mathcal{F} \psi^+, \qquad \psi \in [L_2^+]_2$$

where $G_{M\pm}^{-1}$ are as in (5.28) *and* (5.29).

Remark. It is worth noting that the matrix function G_M belongs to the Wiener algebra $[W]_{2\times 2}$ and thus the generalized canonical factorization of G_M (5.25) is actually a canonical Wiener–Hopf factorization, i.e., $G_{M+}^{\pm 1} \in [W^+]_{2\times 2}$ and $G_{M-}^{\pm 1} \in [W^-]_{2\times 2}$.

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